THE
MATHEMATICAL GAZETTE.

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DISCUSSION ON REFORM IN THE TEACHING OF MATHEMATICS.

THE Annual Meeting of the Mathematical Association was held at King's College, London, on Saturday, January 18th. In the absence of the President, Mr. J. F. Moulton, K.C., the chair was taken by Professor E. M. Minchin, Vice-President. Professor Henrici wrote regretting that he had been prevented from attending. There were 29 members present. The subject discussed, "Reform in the Teaching of Mathematics," was introduced by Professor A. Lodge.

The Chairman, in opening the meeting, said they were prepared for as good an exhibition of Donnybrook Fair as could be given, for there were sure to be many teachers of mathematics present who would not agree with some of the points discussed by Professor Lodge. The fundamental question in the reform of mathematical teaching was the reform of geometry.

Professor Lodge then read his introductory paper.

MR. PRESIDENT, LADIES, AND GENTLEMEN,

The subject of my paper needs no apology before such an audience as this, for the Mathematical Association exists for the purpose of inaugurating and furthering improvements in mathematical teaching. The special object in bringing the whole question forward now is to enable us to co-operate with the British Association Committee formed for the same purpose at the Glasgow meeting last year.

Many teachers have been for a long time aware that the teaching of geometry in this country was suffering from its being based on a fixed ancient model, which, however excellent, was not in many respects satisfactory as a text-book for beginners. Hence the formation of the A.I.G.T., of which Association we are the lineal descendants. The efforts of the Association were, however, powerless to make any appreciable effect on the action of the great examining bodies in the country, and without their co-operation much progress was not possible.

Now, however, with the powerful leverage of the British Association to assist us, we may confidently look for real and lasting progress.

The best method of teaching geometry will no doubt be the question which will require most attention, as that is a matter in which both teachers and examiners must move together if at all. But there are points in which

improvement might be asked for in connection with arithmetic and algebra. I wish chiefly to consider the geometrical question, so I will touch on only a few points. For example, men come up to engineering colleges who are slow and inaccurate in computation, who do not know the contracted methods of multiplication and division, who are as likely as not to put the decimal point in the wrong place. We want boys taught to be ready and rapid computers, to be able to make rough checks on their own work so as to avoid gross errors, to cultivate common sense in connection with problems, and to be in the habit of verifying answers.

This applies to algebra as well as to arithmetic, even more so to algebra perhaps. A boy will perform an absurdity in algebra, quite unconscious of its absurdity, which he would never think of perpetrating in arithmetic. Thus we often find a student writing $\frac{1}{a}+\frac{1}{b}=\frac{1}{a+b}$ who sees at once the absurdity of $\frac{1}{2}+\frac{1}{3}=\frac{1}{5}$. His algebraic work should grow out of arithmetic, and continually be brought back into touch with it by numerical checks.

He should also be taught to multiply and divide rapidly by the method of detached coefficients, and be drilled in short division (synthetic division) in the case of simple divisors.

Then, again, arithmetical work can be so directed as to prepare the boy for geometry, by actual measurement and calculation in connection with geometrical figures, both plane and solid. The ratio of the circumference of a circle to its diameter could be verified by measurements on a cylinder. The area of a rectangle, parallelogram, triangle, etc., should be numerically worked in connection with scale drawings or actual solids, and verified by squared paper. Volumes of actual solids should be calculated from the pupil's own measurements. The eye might be trained to guess areas and volumes and the guesses be verified by actual measurement and calculation.

The algebraic formula $(a+b)(c+d)=ac+ad+bc+bd$ could be illustrated by rectangles drawn on squared paper. A very pretty illustration of this formula is afforded by the duodecimal multiplication of feet and inches by feet and inches, the algebraic answer coming out in square feet, inch-feet, and square inches.

The pupil should learn at an early stage to measure angles in degrees, and to learn by experiment such things as that the angles of a triangle add up to two right angles. The angles of various triangles could be estimated by eye and then measured. The notions of complementary and supplementary angles could be easily introduced, and the angles connected with a transversal crossing parallel lines be considered. Herbert Spencer's Inventional Geometry in the hands of a good teacher would be invaluable in stimulating the energies and intellect of a junior class.

When the pupils have thoroughly grasped elementary geometrical notions, and been practised in measurements and calculations, then they would be ready for a course of deductive geometry.

And here it must be remembered that the pupil's mental equipment is chiefly arithmetic and algebra, and his geometry should be built on these notions as much as possible, instead of being carefully divorced from them as is done in so many text-books. In Euclid's time arithmetical work was abstruse and difficult; in fact, Euclid approached arithmetic through geometry. We have to do the reverse, and we ought to do it wholeheartedly and without reserve.

I believe we could not do better at the outset than adopt some French text-book as our model. The Americans have done so already.

The chief points in the French text-books are:

(1) The more orderly arrangement of propositions.

(2) The entire separation of theorems from problems of construction, hypothetical constructions being used in proving a theorem.

(3) The closer association of a proposition and its converse when both are true.
(4) The adoption of arithmetical notions and algebraic processes.
(5) The early introduction of simple loci.
(6) Insistance on accurate figures drawn by accurate and practical processes.
(7) Practice in exercises from the very beginning.

Mr. Greenstreet suggests that I should also add:

(8) Attention paid to the various phases of a theorem as the figure changes, and (as the student progresses) to the easier forms of generalization.

The greater part of these improvements could be adopted at once, provided the sanction of the great examining bodies can be obtained. The first is the one which presents the greatest difficulties, but many of these difficulties would disappear with the sanction of No. 2.

Of the importance of a rearrangement there can be no doubt.

For example, the propositions that

Two triangles are equal if two sides and the included angle in one are equal to two sides and the included angle in the other, and that *two triangles are equal if two angles and the included side in one are equal to two angles and the included side in the other*,

are obviously related propositions.

In the French text-books they are juxtaposed, both being proved by superposition. Also I. 24, 25, being obviously related to I. 4, are made to immediately follow it in such of the French books as define a straight line to be the shortest distance between two points.

Again, the theorems relating to angles, such as I. 13, 14, 15 should come before triangles, as they do in the French books.

The whole subject of rearrangement is too vast to be treated in the course of a paper—it must be settled by a committee, or at least its most important features would have to be so settled; and the sanction of the great Examining Boards must be obtained.

With regard to the fourth point above, there is nothing in the French geometries corresponding to the greater part of Book II.; being rendered unnecessary by the arithmetical and algebraic treatment of areas. They could be taken as exercises in areas. The only theorems to be retained are II. 12, 13, which can thus immediately follow I. 47, 48, with which they are obviously connected.

[Euc. II. 11, 14 are merely particular cases of the graphic solution of quadratic equations by the method given in Cremona's *Geometry*, which should I think be more generally known and taught. However, these are problems, and therefore not in the category to which we are now attending. So also with the whole of Book IV.]

The French treatment of areas forms a simple introduction to Book VI. They prove that the measure of the area of a parallelogram is the product of its base and height, and that the area of a triangle equals half this product. Hence triangles (or parallelograms) on equal bases have areas proportional to their heights; and those of the same height have areas proportional to their bases.

There is one other suggestion which I should like to make, and that is that the student should be encouraged in anticipating theorems, so as to have his mind prepared for a new theorem before being set to formally prove it; and he should test the truth of new theorems experimentally, using accurately drawn figures.

In conclusion, I would urge on all those who are convinced that reform in geometrical teaching on some such lines as I have indicated is urgent and imperative, that they should not rest content until some at least of the

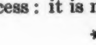
reforms are sanctioned by the great public examining bodies, and I think this meeting should not conclude without appointing a strong committee to co-operate with the British Association Committee and assist it in every way possible. That Committee has already had a valuable communication* signed by 23 schoolmasters, and has asked me to express to this Association its request for the fullest co-operation and advice.

The Chairman said the main point in the question of the teaching of geometry was the problem of dealing with the great waste of time that takes place with the subject. For some reason or another the boy who begins to learn geometry at nine makes hardly any progress by the time he is 14. He goes probably to a preparatory school at nine, and stays there till about 14, when he goes to one of the public schools. During that time he may succeed in getting through two books of Euclid, and some clever boys get over three. That is a very miserable amount of work for five years. What is the cause of it? He had come to the conclusion that the cause was the adoption of Euclid's language and method. The schoolboy is not taught geometry; he is taught to remember the words of Euclid. Mr. Lodge emphasised the teaching of arithmetic and algebra and their use in the teaching of geometry. There are people who say you must not teach arithmetic or algebra in connection with geometry. A few weeks ago he examined a book in which the principle was laid down that those who are doing Euclid may be allowed to save space when speaking of the square on the line AB by writing AB^2, but it was merely as a notation for "the square on the line AB," and it was stated that as soon as the boy conceived that it meant any arithmetical quantity, its use was to be immediately stopped. The boy might write "rectangle AB, CD," but the moment he got an arithmetical notion from that, the notion was to be driven out of him. That idea is probably responsible for the whole waste of time that takes place in the teaching of geometry.

Professor Lodge recently showed him (the speaker) a little Belgian book for the secondary teaching of young girls in Belgium. It was a course of one year's teaching. There is more than six books of Euclid in it, and all through the algebraical and arithmetical notation is used; hence the rapid progress.

There must be several present that afternoon who had had experience of the public examinations: those who have not would, he thought, not believe him if he told them of some of the gross absurdities that examiners in Euclid have to deal with. He would just mention two cases to show them the kind of intellectual process that the learning of Euclid means for the ordinary schoolboy.

At one of the public examinations one boy who came from a very great public school was answering the question, "What are the cases enumerated in Euclid in which two triangles are identical?" The answer expected was: "The 4th, 8th, and 26th propositions of the 1st book"; but this ingenious young gentleman gave 13 cases in the 1st book of Euclid, and he made some of them up in this way: he said, "If there are two triangles, ABC, DEF, and the sides AB, AC of one are equal to DE, DF of the other and the included angles are equal, then the triangles are equal. Also, if AB, BC of the one are equal to DE, EF of the other and the included angles are equal, then the triangles are equal," and so on for 13 cases. Now, such an answer does not indicate an intellectual process.

At the same examination a boy was required to draw a perpendicular to a given line at a given point. He said, "Let AB be the given line and C the given point. Draw CD 

Then shall CD be perpendicular to AB; for, if not, let CE be drawn perpendicular to AB," etc. Now, that again is not an intellectual process: it is nothing but the repetition of a mere jargon.

* *V.* p. 143.

He could give other instances of erratic thought as well as in geometry, but no other branch of mathematics was so full of humorous episodes. One story he might relate seemed to convey a reflection on his own ability to teach statics. He asked a student to define a funicular polygon of a number of forces. The definition he obtained was, "A funicular polygon is a kind of polygon very difficult to draw, and impossible to understand." (Laughter.)

What Mr. Lodge had said about graphic solutions was worthy of attention. These should, in his opinion, accompany every branch of mathematics, and not be restricted to geometry: he hoped that graphic methods of solution would be generally adopted.

That concluded his remarks about the teaching of geometry. There was another question, the reform of teaching of other branches of mathematics—*e.g.*, dynamics and hydrostatics. Some of those present had to teach dynamics as well as geometry. There had been two methods of teaching dynamics in vogue. One was to begin with the teaching of statics and then to pass on to the teaching of kinetics, or what used in the old days to be called dynamics. Thomson and Tait many years ago advocated a different plan. They made no distinction. Force was one and the same thing in both subjects. From the former method the student got an idea that there were two different kinds of force: it was difficult to get this idea out later. Some people tried to make no distinction; they taught the two subjects together. He had recently done it himself, and he found that, taking the two subjects together *ab initio*, more rapid progress was made and sounder ideas inculcated in dynamics than under the old method. With regard to the way in which this subject is to be treated, he thought that those teaching it ought to remember that in the whole subject of statics there are just *two* fundamental facts, which are: First, that if you have any number of vectors, the resultant of these vectors has the same component along any line as the vectors themselves; and, secondly, that the moment of the resultant vector about any axis is equal to the sum of the moments of the vectors about that axis. That ought to be kept prominently before the student; and its importance becomes more pronounced in the most advanced parts of kinetics.

With regard to hydrostatics, he supposed the subject is taught in all the schools; yet he found with the students who came up to the Engineering College at Cooper's Hill that it was impossible to get any accurate hydrostatical notions from them. He sometimes asked them what was the number of cubic inches contained in so many lbs. of a given metal, the specific gravity of which was given in the table of specific gravities provided. Now the students knew that $W = Vs$: they had learnt this formula; and when asked, "How many cubic inches are there in 10 lb. of platinum?" they put $10 = 22V$, 22 being given as the specific gravity of platinum in the table. He (the Professor) asked them: "Is this $\frac{10}{22}$ cubic yards, or miles, or what?" They often reply "Feet," and on being asked the reason say, "Because a cubic foot goes so well with a pound." (Laughter.) There is very great difficulty in getting accurate results in any subjects from students whose attention has not been riveted on the fundamental principles of that subject.

He passed on to kinetics. He supposed they would all say that one of the most difficult subjects was the subject of rigid dynamics. In that subject, if one took up a text-book and opened it at almost any point, one would probably see large clouds of symbols which are quite unnecessary. In the deduction of the great dynamical principles, of which there are three, the principles should be deduced directly from Newton's second and third axioms, and one does not require to write down a single symbol. It follows from the second axiom, the internal forces being equal and opposite in the same right line,

that the resultant acceleration of each particle coincides with the resultant of all the forces internal and external acting on that particle. Now, making use of the two fundamental properties of vectors above mentioned, we have at once the principle of the motion of the centre of mass, and also the principle of the moment of the momentum of any system about any axis. You have then at once an explanation of what one supposes to be one of the most wonderful things ever seen, the motion of the gyroscope.

There is another thing: we have not yet succeeded in eliminating the principle of D'Alembert from our books. We ought ruthlessly to cut it out of our treatises on dynamics. This principle of D'Alembert is contained in the 2nd and 3rd axioms of Newton. The expression that D'Alembert gave it was one in which he introduced the notion of fictitious forces, and one result of this is the fallacious conception of centrifugal force. This he (the speaker) considered to be one of the great physical fallacies. Most people think centrifugal force to be a force tending to drive a body revolving round a centre away from that centre. There is no such force; and it is this principle of D'Alembert which is responsible for the fallacy.

Finally, this Association might possibly do something towards reform of the nomenclature of dynamics. The British Association appointed a committee for dealing with the nomenclature of physics. A similar committee would seem to be required for revising the nomenclature of applied mathematics. Thus, we speak about the *angular momentum* of a system about an axis. Angular momentum is no more momentum than chalk is cheese: it is in reality *moment of momentum.* Another expression is the term *moment of inertia.* What is the moment of inertia of a mere area, which has no mass and no inertia? A conception to which he found students take kindly is the conception of the "mean square of distance from any axis." The radius of gyration is simply the square root of the mean square of distance of a body, area, or volume from an axis.

The necessity for a reform of nomenclature is great in dynamics and in physics too. The physicists seem to be quite happy with such terms as "electromotive force" and "magnetomotive force." Now electromotive force and magnetomotive force ought to mean some kinds of Force. Each term means nothing of the kind. This association should take up the question of reforming other branches of mathematics. It is in the interests of young boys that it should deal with the reform of geometrical teaching: we want to be able to produce some valuable result at the end of five years' teaching instead of the wretched results attained by teaching Euclid's very words and order.

Finally, he would like to ask the question, "What is a Circle?" In an American book and in an English book he had seen it described as the *area* inside what he had been in the habit of calling a circle. The French distinguish between a circle and the circumference. Euclid did not do so. Was the circle then a curve or an area? The difficulty was in the preposition—*on* a circle, or *in* a circle. Euclid, though he defines it as an *area*, uses it as a curve.[1]

Professor Hudson confined himself to the teaching of geometry. He conceived there were two main things to reform in the teaching of geometry. First, to induce teachers to prepare for the teaching of reasoned geometry by a preparation such as that given in Spencer's book. The pupil should become aware of the *facts* of geometry before reasoning about those facts. He was not so much inclined to complain of the slowness in the initial stage to which their chairman had referred. It does require time to lay and settle a good foundation. Slowness in the early stages is compensated for by rapidity afterwards. On this teachers should hold their

[[1] For this point in American and British Geometries v. *Gazette*, p. 118, l. 21 up, and p. 119, l. 9 up. W. J. G.]

own firmly against the parents. Time is *not* wasted though one may not get to the end of the first book in the first year.

The second main point for reform is that of the acceptance of propositions learnt by heart. That goes on to a terrible extent. Pupils can succeed in passing examinations by means of it. Now, it should not be possible to pass an examination in geometry by the mere reproduction of propositions learnt by heart. No alteration in the text-book will bring about a reform if that is to be allowed to continue. If the French or American books were used, and the pupils learnt by heart, and were allowed to pass examinations thereby, we should have all the same trouble as now over again. He agreed with nearly all that Professor Lodge said, but perhaps not with all he *meant* with regard to the use of algebra in geometry. He was strongly of opinion that the second mathematical subject to be taken up was geometry. Arithmetic, geometry, algebra is the correct order. The child does something of arithmetic at the age of two or three, and some progress, slight though it may be, is made in the subject. Geometry then may be taken at from 7 to 9, but algebra could scarcely be begun till a couple of years later. It requires a more mature mind than the other subjects. He would be sorry if a reform in geometry were to drive out the subject as a distinct branch by itself. The two subjects can be carried on side by side, and made to illustrate one another, if the pupils know algebra; but that algebraical proof should be substituted would be very unfortunate and unnatural in the early stages.

Some of his (the speaker's) pupils, who were young ladies, having no notion of algebra, had derived great benefit from pursuing the study of geometry, and doing it well. He would be sorry if any change could render that impossible.

There were various minor points to which he would wish to refer. Euclid's definition of an angle should be altered, and the modern definition adopted in its place. In the third book he thought the doctrine of limits must not be evaded. The modern definition of a tangent is implied in one of Euclid's propositions. Another notion Professor Lodge had referred to was the notion of a "locus." That notion should be introduced early, as should the notion of "symmetry." In the second book, the new notion of "sense" along a line might be treated so as to be understood by a person ignorant of algebra, and, therefore, ignorant of the idea of a "negative" quantity. He wished this treatment of the thing in geometry to be preparatory to the idea of "minus." A great general principle is that teaching should all be anticipatory, and clear the way for some after teaching. That is illustrated by the doctrine of "sense" along a line preparing for the conception of a negative quantity, the definition of a tangent preparing for subsequent fuller treatment of the doctrine of limits, etc. Let our early teaching prepare the way for the subjects which have to come after. They must try to induce teachers to begin with the actual handling of solids. Plane geometry cannot be begun before the age of seven. It is dangerous to put compasses in the hands of very young people.

Professor Hill said that all teachers of mathematics should be grateful to Professor Perry for bringing the subject of the teaching of elementary mathematics before the British Association, and to Professor Lodge for asking the Mathematical Association to discuss it. The concrete representation of problems, wherever possible, which Professor Perry ably advocated, was of the greatest service. Most teachers of mathematics cannot help admitting that there must be something wrong in the teaching of that subject when the results of that teaching are so poor. In seeking for the cause of the failure, the speaker had come to the conclusion that it was to be found in the fact that too great a strain was placed upon the memory. From the very beginning of the subject too many rules were taught and too

little attention was paid to principles. This was a matter upon which the speaker was at direct issue with Professor Perry, and, indeed, there were some passages in Professor Perry's treatise on Practical Mathematics which seemed to show that Professor Perry was at issue with himself. For example while on the one hand on page 120, he says, "Rules in mensuration ought to be stated as formulae, and proved if the proofs are easy, as part of the geometrical work," he says on the other hand (v. p. 22), "Have you not noticed that a great man has only a few simple principles on which to regulate all his actions? A great engineer keeps in his head just a few simple methods of calculation." These two passages from Professor Perry's book state very clearly two different modes of teaching, the one resting on *rules* learned by heart, the other on *principles* and *methods*.

Unfortunately, Professor Perry seems to incline to the method of teaching by rules. He says in his address to the educational science section of the British Association, "Why should not a boy assume the truth of many propositions of the first four books of Euclid, letting him accept their truth, partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra. Letting him assume the sixth book to be axiomatic."

It is this method of teaching by rules that is responsible for the failure of the majority of young people to determine with accuracy the position of the decimal point in a quotient.

Professor Lodge seemed to the speaker to lay too great stress upon the acquisition of rapidity in calculation. Useful as the faculty is, it is not sound policy to sacrifice a thorough comprehension of principles to its acquisition. The main thing in the present circumstances, however, was to develop a constructive policy, and this was perhaps more necessary in the case of the teaching of geometry than in other subjects. Euclid's order is a cause of great difficulty. He seems to have first obtained his propositions, and then after much reflection to have arranged them in groups without reference to the order of discovery. The speaker was once present at a prize distribution and heard the chairman, after first disclaiming all recollection of his mathematics, go on to say that there was one proposition which had left a great impression on his mind of the magnitude of Euclid's genius. This was the construction for an isosceles triangle, in which each of the base angles is double the vertical angle (Euc. IV. 10). On thinking the matter over, it seemed to the speaker that the probable course of ideas was this: Euclid started with the idea of constructing a regular pentagon. It at once became evident that the construction depended on the construction of an isosceles triangle of the kind mentioned above. To construct this, Euclid examined the relation between the segments of the side of any triangle made by the bisector of the angle at the opposite vertex (Euc. VI. 3). Having obtained this, he reduced the problem of the construction of the isosceles triangle required to the cutting of a straight line into two parts, such that the rectangle contained by the whole line and one part is equal to the square on the other part, and this he put into the second book, where it stands as the eleventh proposition.

This is perhaps one of the most striking examples of the artificiality of Euclid's order, though there are many others, some of which have already been mentioned by previous speakers. The real question is, What should now be done? The speaker believed that it was absolutely necessary to re-arrange the treatment of the subject in a more natural order, and that this order should be discovered by experiment. He had himself attempted to teach the subject to a child, and he was surprised at the amount of work that could be done with a pencil, a ruler, a pair of scissors, and a piece of paper. For example, he had cut out the angles of a triangle and put them together, and asked the child, "How much have you?"

The child replied "Half a 'round.'" He then took a quadrilateral and cut out the angles and put them together. The child called the result a whole "round." And so on with a pentagon the angles formed a round and a half. The next day the speaker took a triangle and produced the sides, and cut out the external angles. Putting these together the child said, "You have now a whole 'round,' the same as you had yesterday for a quadrilateral." The next step taken was to put together the external angles of a quadrilateral. Whilst this was being done, the child said, "You will get more now than for the triangle," thus showing that he had formed the idea that the sum of the angles must increase with the number of sides of the figure. When it turned out to be only a "round," the child asked that a five-sided figure should next be taken, and when the same result was reached, he asked that a ten-sided figure might be tried. When this again gave only a "round," the child seemed to see that the result would be always the same; and in this way the truth of the proposition was stamped upon his mind before he was able to understand, or even read, one of Euclid's demonstrations. Experimenting in this way, the speaker believed that it would be quite possible to discover a more natural order than Euclid's, which would present far less difficulty to beginners.

Before passing away from the subject of Euclid, the speaker wished to say something about the fifth book of Euclid. Professor Perry said, in his address to the British Association, that the whole of the fifth book of Euclid should be given by simple algebra. And there had recently appeared a letter in *Nature*, signed by 23 mathematical teachers, stating that no one now teaches the fifth book of Euclid. The speaker was willing to admit that this book, which, with the Logic of Aristotle, Professor De Morgan described as the two most unexceptionable and unassailable treatises ever written, was unsuitable for elementary teaching. For this there was a definite reason, viz., that Euclid uses his test for distinguishing between *unequal* ratios to prove properties of *equal* ratios. Now it is evident that if Euclid's test (Euc. V. Def. 7) for *equal* ratios (Euc. V. Def. 5) is a good and sound one, it should be possible to deduce from it all properties of equal ratios, without using the test for unequal ratios. The neglect to do this makes Euclid's proofs artificial and therefore difficult. When the whole argument is made to depend on his test for *equal* ratios alone, the whole of the difficulty of the book is swept away. The great merit of the book is that the treatment is applicable both to commensurable and incommensurable magnitudes. The algebraic treatment, of which Professor Perry speaks, is applicable only to commensurable magnitudes, and this, in the opinion of the speaker, should be given to boys as soon as they reach Euc. I. 35, 37 [parallelograms (triangles) on the same base, and between the same parallels, are equal to one another], and they then get for commensurable bases Euc. VI. 1 [triangles and parallelograms of the same altitude are to one another as their bases]. In like manner Euc. VI. 33 is reached through Euc. III. 26.

But this mode of developing the subject would not render it unnecessary to provide a place for the treatment of incommensurables at a not very advanced stage of instruction, and to that treatment there was no better introduction than Euclid's fifth book, after it had been modified in the manner indicated above. Incommensurables exist and must be dealt with, and in the opinion of the speaker they should be dealt with before the calculus is commenced, if exact ideas of the theory of limits are ever to be attained.

Mr. F. E. Marshall said it was some 14 or 15 years since he was last present. For more than 30 years he had been gaining experience as to what mathematics boys ought to have done between the ages of nine and fourteen. There were several things in connection with their subject to which he would like to refer. First of all, there was the exceeding badness of the Euclid teaching given to boys at that stage. They ought to recognize

the fact that boys are then taught by people who in many cases have had no mathematical training, whether they be assistant masters in the preparatory schools or governesses at home. He could not help thinking that to circulate a notice, describing the way in which they wanted the subject to be approached, might be productive of good. The thing that most wants saying to those elementary teachers of geometry is, "For goodness sake don't let your students see a text-book." He believed that much of the failure in geometrical teaching of which they complained to-day was due to the use of text-books and the want of some preparation in geometrical drawing. He thought, with Prof. Hudson, that there was a danger of being too numerical in the latter. Teaching without a book, the teacher could watch the pupil's mind as it grasped each new point put before him, with full profit to the pupil and a pleasure to the teacher, in great contrast with the profitless tedium of testing in writing what the pupil had learnt—and perhaps not half understood—from a text-book. In *vivâ voce* the pupil approaches a new theorem by the method of analysis—the natural method of discovery—and afterwards throws his proof into Euclid's synthetical form. He pleaded that some attention might be given to the teaching of arithmetic. He felt much might be got from attention to two or three simple things. They should try to get the elementary teachers to deal thoroughly with the decimal notation. They should go straight from decimal integers to decimal fractions, and not, as they so frequently do, introduce decimal fractions as a distinct subject *after* vulgar fractions. They ought to prepare the way for later teaching and the use of approximate methods. Thus, in multiplication, they should always multiply first by the highest digit. The pupils should be trained to see the working of the metric system, not merely the conversion from that system to another. He considered the teaching of rapidity of computation to be one of the most valuable things they did. By making a boy do the simplest processes with extreme rapidity, his power of fixing his attention is very quickly improved. His mind gives a quicker response to any external stimulus. With regard to algebra, as Professor Hudson says, it should be *third*, and the foundations for it should be soundly laid in the teaching of arithmetic. They, in public schools, received boys who professed to solve quadratic equations, and yet were unable to work out correctly simple questions in vulgar or decimal fractions. It seemed to him that they should try to formulate their desires in the matter of geometry, and then leave the matter of a text-book to come later. It took such a long time to get a text-book or syllabus agreed upon. Could they not get the examining bodies to sanction a little change, and so proceed little by little? If they could first get the examining bodies to cut out a few of the things they all objected to, they could then ask for more. He felt there was a great danger in asking too much at once.

Mr. E. M. Langley agreed cordially with the previous speakers so far as they agreed with one another. He especially agreed that they ought not to wait for big things, though of course he hoped they might influence examiners. They ought to set to work on experimental courses that would help to make boys see what their mathematics meant. They had, as the A.I.G.T., made a great mistake in asking that no text-book might be authorized—it was too revolutionary a step for the authorities. One of the chief difficulties would come with the teachers, especially with the junior teachers. It would be no good saying to them, "Go and teach experimental geometry straight off." They would say at once, "Very well, how are we to do it?" As to text-books, of course, the boys would be able to do without the text-book if the teacher could. (Laughter.) There could be no doubt that every branch of mathematics should be illustrated as far as possible from other branches. There are no water-tight compartments in mathematics. He used to be stared at when he first asked a boy solving equations, "What

axiom are you using there?" As if axioms were only used in Euclid. Boys should certainly be encouraged to test their answers. He did not know whether they knew Professor Perry's story of the postage stamp. A Cambridge undergraduate was asked how many postage stamps would be required to cover the walls of a room—the Senate house, perhaps—and after a great deal of work the answer 1, with a decimal running to some twenty or more places, was produced. (Laughter.) He had one of his own almost as good. When he (the speaker) started teaching the metric system he held up a boxwood metre and, explaining to the class that a kilometre was a thousand of those, he asked how long a man would take to get from his house to a station a kilometre away, walking at the rate of four miles an hour. The answers varied from six weeks down to 1·1057 of a minute.

Professor Lodge had shown them a case of the graphic solution of a quadratic equation from Cremona. That had been given in Leslie's *Geometry*, where it was said to have been suggested by a student of Leslie's—*Mr. Thomas Carlyle, an ingenious young mathematician*—so that the solution shown them on the board was invented by Thomas Carlyle.

He did not think problems of construction should be entirely separated from the theorems. They should teach the student a theorem and then say to him, "Now, what good can you get from it?" They ought to let boys try things with one little difficulty on their own account. As for rapidity of computation, computation should be rapid, otherwise its value is gone; no advance would have been made over quite early times. He (the speaker) had been shown a little pamphlet which contained an Old English translation of some precepts on arithmetic that had originally been given in Latin verse. He had not troubled about the Latin verse, but he had been able to make something of the translation. Addition and subtraction were taken, and then the author goes on to "duplation," or multiplying by two. Next he takes in division by two, or "dimidiation." As an intellectual exercise this method is excellent, as they would soon be convinced if, beginning *at the right hand side* and working towards the left, they tried to divide any number by two. It will give still greater intellectual training if they divided by three or some higher number in the same way, but the British parent has a right to expect his boy to be able to calculate rapidly and well. He considered it a great thing in teaching youngsters to have figures outlined with movable rods, so as to get the idea of moving and varying quantity into them. Various little helps may be given to the pupils. You may get the idea of the three angles of a triangle being together equal to two right angles by making a paper triangle, folding in the three angles, and then you actually see they make two right angles.

[The speaker illustrated these by brown paper figures, jointed rods, etc.]

Dr. F. S. Macaulay said the invitation which had been given to Prof. Lodge was from the committee formed at the Glasgow meeting of the British Association. Although the Council of the Mathematical Association had not yet formed a committee, they had started to form one. The idea that the Council had was that they would like to get a large number of schools near London represented upon the committee, so that they might have as members those who would be able to attend the meetings. The committee would be formed mainly of those who were engaged in some form of elementary teaching.

The subject was a very difficult one. They could not hope to revolutionize the teaching of mathematics. If they could get some reforms they ought to be satisfied, and they ought not to be disappointed if their efforts did not realize what some people seemed to expect.

He, personally, was most interested in geometry. He did not altogether agree with what had been said. A document had been issued, signed by 23 masters in public schools, which had been communicated to the Committee

of the British Association, in which a good many valuable suggestions were made. The first suggestion was one on which they all seemed to be agreed :—"That the most practical direction of reform is towards a wide extension of accurate drawing and measuring." Directly we get beyond that, however, we get on to dangerous ground ; we are advised to omit such propositions of Euclid as do not "serve as landmarks." That is not clear. Again :—" We can well dispense with many propositions in the first book." It seemed to him (the speaker) that, with the exception of constructions, all the propositions in the first book were landmarks. "The third book is both easy and interesting." He agreed, but it is not so easy and interesting as the first book. The second book certainly is very difficult, and if it could be dispensed with altogether so much the better. "Euclid proves several propositions, whose truth is obvious to all but the most stupid, or the most intellectual." He did not know what this meant, for the propositions which seemed to be indicated were not those which were very difficult. They were, then, to trust to the boy himself to say whether a proposition was evident or not, and to treat him as if he were a geometer of the highest order.

He considered the fifth book of Euclid specially important because it does deal with incommensurables. He believed that the methods in the fifth book afforded the most useful and the most easy way of introducing boys to the ideas of incommensurables. The fact that incommensurables can be dealt with in geometry as well as in algebra gave them a choice as to which way should be used first. It seemed to him that the fifth book is really one of the most important parts of Euclid, if not *the* most important part.

Mr. C. Godfrey agreed as to the importance of practical geometry. At the same time riders of the ordinary type formed a valuable exercise. A greater number of boys should be able to attain to the power of doing riders. But this would always be a slow business. He, as a practical teacher, was constantly met with the impossibility of finding time. He laid the blame upon the necessity of teaching a great number of propositions. They would gain by omitting a fair number of the propositions. Restriction in the use of compasses was a restriction they were not compelled to keep up. Was not proposition I. 7 rather difficult and unnecessary? When they came to areas again, several propositions might be cut out. The use of the ordinary draughtsman's rule would get them out of several of the difficulties to which Prof. Lodge had referred. They might, too, adopt the French method of taking the third book before the second. In Book III. there were many worrying propositions. He always found a difficulty in proving that two circles cannot meet in four points. Mr. H. M. Taylor, indeed, confessed that it is impossible to draw figures which will be satisfactory in propositions of that kind. With regard to the second book, he was inclined to treat it algebraically. He admitted that he had tried it so, and had not been quite satisfied with the experiment. He found the boys' minds were not sufficiently developed at that stage, but anticipated that this difficulty would not arise if the Book II. were taken after Book III. There was a strong argument for treating the second book algebraically. In geometrical conics no one would attempt to deal a problem without algebra. One should not prohibit algebra, though the bulk of the proof might not be algebra. As for the fourth book, he had found the proofs very tedious when he was learning them. In the fifth and sixth books it is clear that simple algebraical or arithmetical treatment would meet the case of commensurables. Need they be strict about incommensurables at that stage? In their algebraic teaching, had they not made such assumptions as that $\sqrt{2} \times \sqrt{3}$ equalled $\sqrt{3} \times \sqrt{2}$? You cannot prove that unless you make some very grave assumptions at an early stage. They did not make a more serious

assumption if they confined their treatment of the sixth book to the commensurable quantities. The time for a more thorough treatment would be during the study of limits and convergent processes. Prof. Lodge's suggestion as to the constant use of squared paper he considered a very good one. In giving problems on areas a difficulty might occur as to the form in which to put the exercise. Suppose it is a trapezium whose area has to be calculated, squared paper might be very useful here. One may give the class the co-ordinates of the angular points, and from those the pupils may draw the trapezium. A graduated series of exercises can be set in this way.

Then there was the question as to the need of practising manipulation. A good teacher could give his class valuable practice orally. But if skill in manipulation is proposed as the object to be aimed at, many teachers would probably set long rows of sums from a book. He thought they should say rather less about facility of manipulation, and that the stress should be laid rather on the understanding of the process.

Mr. T. Wilson recommended the working of riders from the beginning. He would go through the first 48 propositions in the Harpur Euclid, and then proceed to one of the foreign manuals. That of Luvini, an Italian, he believed to be excellent. He did not agree with that speaker who advocated slowness in learning mathematics. He would like pupils to attain more rapidly than now some knowledge of solid geometry. He remembered he himself went right through the second book of Euclid thinking that a rectangle was a right angle. There was this to be said, however, in his favour, he used an old edition of Simson's Euclid, and in it a rectangle was nowhere defined. He did not see how they were to do without text-books. Very clever boys should have access to a good text-book: such boys would go on rapidly without a teacher's help. They all lived upon a sphere, and knew something about geography, and boys found no difficulty in understanding latitude and longitude—the co-ordinates of a sphere defining a point, and somewhat more difficult to understand than plane Cartesian co-ordinates.

He himself had been a pupil of Prof. de Morgan, and wished he had paid more attention to his instruction. De Morgan made the conception of negative quantities, the differentia between arithmetic and algebra. Boys knew the difference between north and south, and from that should have no difficulty in grasping the idea of "plus" and "minus."

Mr. T. J. Garstang confessed to being somewhat in the position of the outside assistant master. A member had expressed the wish to see some evidence of progress in the teaching of mathematics. Well, he happened to be in a school[1] where, in the absence of a governing body, the teaching was not fettered by tradition. Boys and girls were taught in the same classes together; both ladies and gentlemen were engaged to teach them. They would gather from that statement that very little attention had been paid to prejudice. He taught mathematics in this school, but he had approached mathematics from the side of science. Consequently, it seemed to him that experimental methods were obviously the right ones to adopt. When you think how easy it is for a boy to describe a circle with a pair of compasses, cut it out in cardboard, and then weigh it in a scale, you also see that the boy may learn his formula and verify it. So much can be done by such experimental methods, in the way of laying a foundation, on which one can build afterwards. He had been much influenced in his logical methods by Prof. Karl Pearson's *Grammar of Science*. Among other things this author discusses the fundamental axioms of Newton, and deals also with the metaphysical way in which the word "force" has been used. A large part of children's difficulties arises through attempts made to remember

[1] [The school alluded to was Bedales School, Petersfield, Hants, recently removed from Hayward's Heath.]

definitions of words, when these definitions are not at all intelligible to them. Many of them have put down "force" as "that which communicates motion to bodies" without understanding what it means. One can get an idea of "force" sufficient for work, by treating it purely as a quantity in mathematics.

In logic, the training should go on quite gradually, and ought not to impede the children's powers of observation. He had adopted in the higher forms the text-book issued by the Association; in the lower, no text-book at all was used. From the point of view of not using a text-book, he might tell them that no text-books were used for *any* subject in the lowest forms of the school. In modern languages the children learnt to sing easy songs and to talk about common things in the modern tongue, before seeing a single word of print. He considered Prof. Chrystal's *Introduction to Algebra* far ahead of all other elementary text-books; the idea of co-ordinates is there given quite clearly. Co-ordinates occur in the first edition of Wood's *Algebra* about 1831, but drop out on revision by Lund; and it seemed to him that between 1831 and 1898 little or no progress had been made in this direction. In algebra all the work in division ought to be postponed till the boy had learnt the method of detached coefficients. Prof. Chrystal's order is much more instructive, and gets the pupil on more quickly. With regard to examinations, examiners have to try to find questions which have never been set before. They endeavoured in his school to get a boy on to some work in which he would be interested, such as electricity; he hoped to be able to teach the fundamental notions and a working knowledge of the calculus, so that a pupil might use his knowledge in the practical affairs of life when he left school. Several hundred years had passed since Cocker's *Arithmetic*, and yet the questions now set were very much like those there given. What was the good of them? Any boy well taught can do the purely arithmetical part; what he does not know in questions on stocks, etc., is, Why do shares rise or fall? and, What is the meaning of broker's commission? As part of their own work, the boys were taken out for elementary surveying. All had to go through the carpenter's shop, and through the laboratory—boy or girl. When a child had done that, there is no difficulty in teaching him decimals. He has a pair of forceps, the weights of the metric system are before him, varying in size and sometimes of different metals, and he can hardly help putting them in their proper places.

Professor Lodge, in reply, thought the use of squared paper would grow if a proper supply were always at hand.

He had been asked by Mr. Wilson to show what he meant by a hypothetical construction. Take Props. 24 and 25 of Book I. Suppose ABC, ABD are the two triangles, with common side AB, and with $AC=AD$, but the angle CAB greater than DAB. If the bisector AE of the angle CAD is drawn, cutting BC in E, we have $BC=BE+ED>BD$. The truth of this is quite independent of whether the pupil knows how to bisect an angle. If the bisector is drawn the theorem becomes evident. That is what is meant by a hypothetical construction. [*Note.*—In some of the French *Geometries* these propositions immediately follow Euc. I. 4, with which they are evidently connected. This would have been impossible if it were necessary to elaborate a method of constructing a bisector before being entitled to use it.]

Professor Hudson had suggested that the definition of an angle should be the modern one: the rotary one. If you permit this you could prove Euclid I. 32 as early as you please. For if a line, initially coinciding with ACD, revolve about A through the angle A and then about B through the angle B, the angle (BCD) which it now makes with its initial position must be equal to the sum of the angles A and B through which it has revolved. [Professor Hudson had stated that the exterior angles were shown

equal to four right angles by taking three chairs and walking once round them.]

Professor Lodge, resuming, said a reduction in the number of propositions could be made if a proposition and its converse were lumped into one, and a further simplification would be made if constructions were put in a different part of the book from theorems.

With regard to incommensurables, he considered they were not fitted for elementary subjects nor for practical work.

Rapidity of computation became interesting to boys if they were taught to emulate each other at the proper stage. He wished to call their attention to the fact that the Board of Education would issue, free, as many tables of four-figure logarithms as they liked.

He hoped the committee which was being formed would be able to make valuable suggestions to the B.A. Committee. They would be glad to receive suggestions from any members of the Association; and if these were sent to Mr. Pendlebury, no doubt he would see that they came before the committee.

THE PUBLIC SCHOOLS AND THE QUESTION.

To the Committee appointed by the British Association to Report upon the Teaching of Elementary Mathematics.

Gentlemen,—At the invitation of one of your own body, we venture to address to you some remarks on the problems with which you are dealing, from the point of view of teachers in public schools.

As regards geometry, we are of opinion that the most practical direction for reform is towards a wide extension of accurate drawing and measuring in the geometry lesson. This work is found to be easy and to interest boys; while many teachers believe that it leads to a logical habit of mind more gently and naturally than does the sudden introduction of a rigid deductive system.

It is clear that room must be found for this work by some unloading elsewhere. It may be felt convenient to retain Euclid; but perhaps the amount to be memorised might be curtailed by omitting all propositions except such as may serve for landmarks. We can well dispense with many propositions in the first book. The second book, or whatever part of it we may think essential, should be postponed till it is needed for III. 35. The third book is easy and interesting; but Euclid proves several propositions whose truth is obvious to all but the most stupid and the most intellectual. These propositions should be passed over. The fourth book is a collection of pleasant problems for geometrical drawing; and, in many cases, the proofs are tedious and uninstructive. No one teaches Book V. A serious question to be settled is—how are we to introduce proportion? Euclid's treatment is perhaps perfect. But it is clear that a simple arithmetical or algebraical explanation covers everything but the case of incommensurables. Now this case of incommensurables, though in truth the general case, is tacitly passed over in every other field of elementary work. Much of the theory of similar figures is clear to intuition. The subject provides a multitude of easy exercises in arithmetic and geometrical drawing; we run the risk of making it difficult of access by guarding the approaches with this formidable theory of proportion. We wish to suggest that Euclid's theory of proportion is properly part of higher mathematics, and that it shall not in future form part of a course of elementary geometry. To sum up our position with regard to the teaching of geometry, we are of opinion:

1. That the subject should be made arithmetical and practical by the constant use of instruments for drawing and measuring.
2. That a substantial course of such experimental work should precede any attack upon Euclid's text.
3. That a considerable number of Euclid's propositions should be omitted; and in particular
4. That the second book ought to be treated slightly, and postponed till III. 35 is reached.
5. That Euclid's treatment of proportion is unsuitable for elementary work.

Arithmetic might well be simplified by the abolition of a good many rules which are given in text-books. Elaborate exercises in vulgar fractions are dull and of doubtful utility; the same amount of time given to the use of decimals would be better spent. The contracted methods of multiplying and dividing with decimals are probably taught in most schools; when these rules are understood, there is little left to do but to apply them. Four-figure logarithms should be explained and used as soon as possible; a surprising amount of practice is needed before the pupil uses tables with confidence.

It is generally admitted that we have a duty to perform towards the metric system; this is best discharged by providing all boys with a centimetre scale, and giving them exercise in verifying geometrical propositions by measurement. Perhaps we may look forward to a time when an elementary mathematical course will include at least a term's work of such easy experiments in weighing and measuring as are now carried on in many schools under the name of Physics.

Probably it is right to teach square root as an arithmetical rule. It is unsatisfactory to deal with surds unless they can be evaluated, and the process of working out a square root to five places provides a telling introduction to a discourse on incommensurables; furthermore it is very convenient to be able to assume a knowledge of square root in teaching graphs. The same rule is needed in dealing with mean proportionals in geometry.

Cube root is harder and should be postponed until it can be studied as a particular case of Horner's method of solving equations approximately.

Passing to algebra, we find that a teacher's chief difficulty is the tendency of his pupils to use their symbols in a mechanical and unintelligent way. A boy may be able to solve equations with great readiness without having even a remote idea of the connection between the number he obtains and the equation he started from. And throughout his work he is inclined to regard algebra as a very arbitrary affair, involving the application of a number of fanciful rules to the letters of the alphabet.

If this diagnosis is accepted, we shall be led naturally to certain conclusions. It will follow that elementary work in algebra should be made to a great extent arithmetical. The pupil should be brought back continually to numerical illustrations of his work. The evaluations of complicated expressions in a, b, and c may of course become wearisome; a better way of giving this very necessary practice is by the tracing of easy graphs. Such an exercise as plotting the graph $y=2x-\frac{x^2}{4}$, provides a series of useful arithmetical examples, which have the advantage of being connected together in an interesting way. Subsequently, curve-tracing gives a valuable interpretation of the solutions of equations. Experience shows that this work is found to be easy and attractive.

With the desire of concentrating the attention of the pupil on the meaning rather than the form of his algebraical work, we shall be led to postpone certain branches of the subject to a somewhat later stage than is usual at present. Long division, the rule for H.C.F., literal equations, and the like,

will be studied at a period when the meaning of algebra has been sufficiently inculcated by arithmetical work. Then, and not till then, will be the time to attend to questions of algebraic form.

But at no early stage can we afford to forget the danger of relapse into mechanical work. For this reason it is much to be wished that examining bodies would agree to lay less stress upon facility of manipulation in algebra. Such facility can generally be attained by practice, but probably at the price of diminished interest and injurious economy of thought. The educational value of the subject is sacrificed to the perfecting of an instrument which in most cases is not destined for use.

To come to particulars, we think that undue weight is often given to such subjects as algebraic fractions and factors. The only types of factors which crop up continually are those of x^2-a^2, $x^2 \pm 2ax+a^2$, and, generally, the quadratic function of x with numerical coefficients.

In most elementary algebra books there is a chapter on Theory of Quadratic Equations, in which a good deal of attention is paid to symmetric functions of roots of quadratics. No further use is to be made of this till the analytical theory of conics is being studied. Might not the theory of quadratics be deferred till it can be dealt with in connection with that of equations of higher degree?

Indices may be treated very slightly. The interpretation of negative and fractional indices must of course precede any attempt to produce logarithms; but when the extension of meaning is grasped, it is not necessary to spend much more time on the subject of indices; we may push on at once to the use of tables.

It will be seen that our recommendations under the head of Algebra are corollaries of two or three simple guiding thoughts; the object in view being, —to discourage mechanical work; the means suggested,—to postpone the more abstract and formal topics and, broadly speaking, to arithmeticise the whole subject.

The omission of part of what is commonly taught will enable the pupil to study, concurrently with Euclid VI., a certain type of diluted trigonometry which is found to be within the power of every sensible boy. He will be told what is the meaning of sine, cosine, and tangent of an acute angle, and will be set to calculate these functions for a few angles by drawing and measurement. He will then be shown where to find the functions tabulated, and his subsequent work for that term will consist largely in the use of instruments, tables, and common-sense. A considerable choice of problems is available at once. He may solve right-angled triangles, work sums on "heights and distances," plot the graphs of functions of angles, and make some progress in the general solution of triangles by dividing the triangle into right-angled triangles. Only two trigonometrical identities should be introduced—$\sin^2\theta+\cos^2\theta=1$, and $\frac{\sin\theta}{\cos\theta}=\tan\theta$. In short, the work should be arithmetic, and not algebra.

Formal algebra cannot be postponed indefinitely; perhaps now will be the time to return to that neglected science. We might introduce here a revision course of algebra, bringing in literal equations, irrational equations, and simultaneous quadratics, illustrated by graphs, partial fractions, and binomial theorem for positive integral index. Side by side with this it ought to be possible to do some easy work in mechanics. Graphical statics may be made very simple; if it is taken up at this stage, it might be well to begin with an experimental verification of the parallelogram of forces, though some teachers prefer to follow the historical order and start from machines and parallel forces. Dynamics is rather more abstract; a first course ought probably to be confined to the dynamics of rectilinear motion.

It is not necessary to discuss any later developments. The plan we have

advocated will have the advantage of bringing the pupil at a comparatively early stage within view of the elements of new subjects. Even if this is effected at the sacrifice of some deftness in handling a, b, and c, one may hope that the gain in interest will be a motive power of sufficient strength to carry the student over the drudgery at a later stage. Some drudgery is inevitable if he is ultimately to make any use of mathematics. But it must be borne in mind that this will not be required of the great majority of boys at a public school.—We beg to remain, Gentlemen, yours faithfully,

G. M. BELL, Winchester; H. H. CHAMPION, Uppingham; H. CRABTREE, Charterhouse; F. W. DOBBS, Eton; C. GODFREY, Winchester; H. T. HOLMES, Merchant Taylors' School; G. H. J. HURST, Eton; C. H. JONES, Uppingham; H. H. KEMBLE, Charterhouse; T. KENSINGTON, Winchester; E. M. LANGLEY, Bedford Modern School; R. LEVETT, King Edward's School, Birmingham; J. W. MARSHALL, Charterhouse; L. MARSHALL, Charterhouse; C. W. PAYNE, Merchant Taylors' School; E. A. PRICE, Winchester; D. S. SHORTO, Rugby; A. W. SIDDONS, Harrow; R. C. SLATER, Charterhouse; H. C. STEEL, Winchester; C. O. TUCKEY, Charterhouse; F. J. WHIPPLE, Merchant Taylors' School.

REVIEWS AND NOTICES.

Choice and Chance, with 1000 Exercises. By Prebendary W. A. WHITWORTH. Fifth Edition, much enlarged. Pp. viii., 342. 6s. 1901. (Deighton, Bell.)

This interesting volume "contains some 45 pages more than its predecessor." "The most important addition in the body of the work is the very far-reaching theorem which enables us to write down at sight the value of such functions as a^3, $a^3\beta^4$, $a\beta\gamma$... when a, β, γ... are the parts into which a given magnitude is divided at random." The exercises at the end are increased to 1000. Prebendary Whitworth's work is too well known to require further comment.

Algebra for Junior Students. By TELFORD VARLEY. Pp. viii., 152. 1s. 1901. (Allman.)

This little book covers "all examinations intended for younger students." It is thoughtfully written and better than most of its class. It is a pity that detached coefficients are not taught as early as possible, and that the solution of quadratics by splitting into factors is not mentioned. When the author states in his preface that the contents may be mastered in two years, we despair of mathematical teaching in this country and heartily sympathise with Professor Perry. There is no reason why the contents should not be mastered in two terms at most.

Elementary Algebra. By ROBERT GRAHAM. Third Edition. Pp. viii., 312 (34). 6s. 1901. (Longmans, Green.)

This edition contains some 28 pages more than in the first edition. The best chapters in this book are those dealing with imaginary quantities, imaginary factors, and "more difficult equations." The author does not teach the use of detached coefficients, but he is "advanced" enough to use determinants in certain equations of three unknowns, though we cannot see why he did not introduce them to the student who is tackling equations with two unknowns. The examples are very carefully graduated.

Woolwich Mathematical Papers for Admission into the Royal Military Academy for the years 1891-1900. Edited by E. J. BROOKSMITH. 6s. 1901. (Macmillan.)

This volume will be found serviceable to Army Classes. It contains the questions in Pure and Applied Mathematics for the years mentioned, together with the answers. Mr. Brooksmith is an Instructor at the Royal Military Academy.

The Tutorial Algebra. Part I., Elementary Course. By R. DEAKIN. Pp. viii., 443. 1901. 3s. 6d. (University Tutorial Press.)

"Suppose $+2\times-3$ or $-2\times+3$. Evidently the product will not be the same in either of these cases as in $+2\times+3$. Therefore we conclude that $+2\times-3=-6$ and $-2\times+3=-6$.... Again suppose -2×-3. This is different from the last two cases, and we conclude that $-2\times-3=+6$.... From these results we can infer the rule of signs." The writer claims in his preface that he has tried to encourage thought from the very beginning of the work. The quotation is a specimen of the way he does it—on p. 26. We do not see that this book is any better than most of its rivals, but for the inclusion of a chapter on graphs. It is well printed and got up.

Practical Mathematics for Beginners. By F. CASTLE. Pp. ix., 314. 2s. 6d. 1901. (Macmillan.)

Elementary Practical Mathematics. By M. T. ORMSBY. Pp. xii., 410. 1900. (Spon.)

These books, of course, owe their existence to the inspiration of Prof. Perry, whose crusade against the present system of teaching Mathematics in our schools is likely to be effectual. Mr. Castle's larger volume on the same subject has already met with a considerable measure of success. The book under notice is designed to cover the ground laid down in the syllabus recently issued by the Board of Education. Within those lines it is excellent. Mr. Ormsby, while also endeavouring to meet the requirements of the same Board, has had in view the needs of more advanced students, especially students of Civil Engineering, who wish to be taught "the additional matter they are likely to require at an early stage, and how to use their knowledge, when acquired, for practical calculations." Mr. Ormsby is a good teacher, as may be seen from the careful way in which most of the demonstrations are reasoned out for the benefit of the private student.

Mathematisches Vokabularium. By F. MÜLLER. Französisch-Deutsch und Deutsch-Französisch. Zweite Hälfte. Pp. xiv., 316. 1901. 11 m. (Teubner, Leipzig.)

We have already drawn the attention of readers of the *Gazette* to the first part of this dictionary. On page xi. of the second part appear some sixty "Nachträge' to the first part, and it does credit to Herr Müller that but three of these are corrections. On the other hand there are some two hundred additions and corrections to the second part on pp. xii.-xiv., and of these a large proportion are "Verbesserungen." It should, however, be noticed that there are fifty more pages in the second part than in its predecessor. This is due to the fact that there are fewer technical terms in French than in German, and also to an increase in the total number of additions and historical references. There are still a few misprints of an obvious nature, such as "proporionale," p. 213. A careful study of selected portions of the text confirms us in the favourable impression produced upon our mind by the first part. Herr Müller reminds us of the saying attributed to the great Littré, "les travaux lexicographiques n'ont point de fin." That no doubt is true, but anyone who has used this dictionary for, say five years, by the extent of reading that implies, will find himself practically independent of such additional matter as it may become necessary to incorporate into the volume during an ordinary lifetime. To the young student this dictionary should prove invaluable.

BOOKS, ETC., RECEIVED.

Annals of Mathematics. Edited by ORMOND STONE and others. Second Series. Vol. III. No. 1. Oct. 1901. pp. 44. 2/- (Longmans, Green).

[*On the Convergence of the Continued Fraction of Gauss and Other Continued Fractions*: Van Vleck. *On the Differentiation of an Infinite Series Term by Term*: Porter. *A note on Geodesic Circles*: Whittemore. *Note on certain functions defined by Infinite Series*: Osgood. *Nim, a game with a complete Mathematical Theory*: Bouton. *On the Groups generated by Two Operators of Order Three, whose product is also of Order Three*: Miller. *On the Invariants of a Quadrangle under the largest sub-group, having a fixed point, of the General Projective Group in the Plane*: Granville.]

Annals of Mathematics. Edited by ORMOND STONE and others. 2nd Series, III. 2. (Harvard University and Longmans.) Jan., 1902. 2/-.

[*Some applications of the method of Abridged Notation*: Bôcher. *On the roots of functions connected by a Linear Recurrent Relation of the Second Order*: Porter. *Space of Constant Curvature*: Woods.]

Leçons d'Algèbre. By Mme. A. SALOMON. pp. 184. 2 fr. 1901 (Nony).

Woolwich Mathematical Papers, 1891-1900. Edited by E. J. BROOKSMITH. 6s. 1901 (Macmillan).

Practical Mathematics for Beginners. By F. CASTLE. pp. ix., 314. 2/6. 1901 (Macmillan).

Text-Book of Practical Solid Geometry, etc. By Capt. E. H. de V. ATKINSON, R.E. 2nd Edition. pp. 124. 7/6. 1901 (Spon).

Nautical Astronomy. By J. H. COLVIN. pp. 127. 2/6 net. 1901 (Spon).

An Elementary Treatise on the Calculus. By G. A. GIBSON. pp. xix., 459. 1901 (Macmillan).

Annuaire pour l'an 1902. Publié par le Bureau des Longitudes. pp. 656 + 185. 1 fr. 50 c. 1902 (Gauthier-Villars).

Geometric Exercises in Paper Folding. By T. SUNDARA ROW. pp. x., 148. 4/6 net. 1901 (Kegan Paul, Open Court).

Culegere de Probleme de Arithmetica, Geometrie, Algebra, şi Trigonometrie. Edited by Messrs. IONESCU, and ŢIŢEICA. pp. viii., 520. 1901 (Göbl, Bucharest).

Spherical Trigonometry. By the late I. TODHUNTER. Revised by J. G. LEATHEM. pp. ix., 275. 7/6. 1901 (Macmillan).

**Grundlinien des Politischen Arithmetik.* By M. KITT. pp. 78 + pp. 39 (tables). 3 marks. 1901 (Teubner).

**Lehrbuch der Combinatorik.* By E. NETTO. pp. 258. 9 marks. 1901 (Teubner).

**Der Naturwissenschaftliche Unterricht in England insbesondere in Physik und Chemie.* By K. T. FISCHER. pp. 94. 3 m. 60 pf. 1901 (Teubner).

**Theorie der Riemann'schen Thetafunktion.* By Dr. ROST. pp. 66. 1901 (Teubner).

**Jahresbericht der Deutschen Math.-Verein.* x Band. 2 Heft. 1 Hälfte. Burkhardt. *Entwickelungen nach oscillierenden Funktionen.* pp. 176. 5m. 60. 1901 (Teubner).

Leçons sur les Séries à Termes Positifs. By EMILE BOREL. pp. 94. 1902 (Gauthier-Villars).

La Géométrie Atomique Rationelle. By J. F. BONNELL. pp. 100. 1902 (Gauthier-Villars).

Theoretical Mechanics. By W. WOOLSEY JOHNSON. p. 434. 1901 (Wiley, Chapman Hall).

Traité de Cinématique Théorique. By H. SICARD, annotated by A. LABROUSSE. pp. 187. 1902 (Gauthier-Villars).

Cours de Mécanique. By P. APPELL. pp. 272. 1902 (Gauthier-Villars).

College Algebra. By J. H. BOYD. 21 + 11 + 787 pp. 1901. Hf. leather. $2. (Scott & Foresman, Chicago.)

Aufgaben zur Differential- und Integralrechnung. By H. DÖLP and E. NETTO. 3 + 216 pp. Cloth. 4 m. 1901 (Ricker, Giessen).

Supplementary Report on Non-Euclidean Geometry. By G. B. HALSTED. [*Science*, N.S. 14; pp. 705-717.]

Linear Groups, with an Exposition of the Galois Field Theory. By L. E. DIXON. 10 + 312 pp. Cloth. 12 m. 1901 (Teubner, Leipzig).

Bulletin of the American Mathematical Society. Edited by F. MORLEY and others. Series 2. Vol. VIII. No. 3. Dec. 1901.

[Modern Methods of Treating Dynamical Problems: *E. W. Brown.* Some Curious Properties of Conics touching the Line Infinity at one of the Circular Points: *E. V. Huntington* and *J. K. Whittemore.*]

Ditto. No. 4. Jan. 1902.

[Note on Mr. G. Pierce's Approximate Construction for π: *E. Lemoine.* Concerning the $\wp(g_2, g_3, Z)$ functions as coordinates in a line complex: *H. F. Stecker.* Abelian Groups conformal with non-Abelian Groups: *G. A. Miller.* The Infinitesimal Generators of certain Parameter Groups: *S. E. Slocum.*]

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